

Irreducible laminations for IWIP Automorphisms of a free product and Centralisers

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Abstract

For every free product decomposition $G = G_1 * \dots * G_q * F_r$, where F_r is a finitely generated free group, of a group G of finite Kurosh rank, we can associate some (relative) outer space \mathcal{O} . In this paper, we develop the theory of (stable) laminations for (relative) irreducible with irreducible powers (IWIP) automorphisms. In particular, we examine the action of $Out(G, \mathcal{O}) \leq Out(G)$ (i.e. the automorphisms which preserve the set of conjugacy classes of G_i 's) on the set of laminations. We generalise the theory of the attractive laminations associated to automorphisms of finitely generated free groups. The strategy is the same as in the classical case (see [1]), but some statements are slightly different because of the factor automorphisms of the G_i 's.

As a corollary, we prove a generalisation of the fact that the centralisers of IWIP automorphisms are virtually cyclic. However, in our statement for the (relative) centraliser of a (relative) IWIP automorphism, the factor automorphisms of G_i 's appear. As a direct corollary, if $Out(G)$ is virtually torsion free and every $Out(G_i)$ is finite, we prove that the centraliser of an IWIP is virtually cyclic. Finally we give an example which shows that we cannot expect that any centraliser of an IWIP is virtually cyclic, as in the free case.

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1 Introduction

Let G be a group which splits as a free product $G = G_1 * \dots * G_q * F_r$. Guirardel and Levitt in [11] constructed an outer space relative to any free product decomposition for a f.g. group and later Francaviglia and Martino in [10] noticed that the outer space $\mathcal{O} = \mathcal{O}(G, (G_i)_{i=1}^q, F_r)$ can be constructed for any group G of finite Kurosh rank. Let $Out(G, \mathcal{O})$ be the subgroup of $Out(G)$, which consists of the automorphisms which preserve the conjugacy classes of G_i 's (note that in the case of the Grushko decomposition, $Out(G) = Out(G, \mathcal{O})$). We could define the notion of irreducibility using representatives of automorphisms between the elements of \mathcal{O} which leave invariant subgraphs, but here it is less complicated to use the notion of free factor systems. More specifically, we say that an element $\phi \in Out(G, \mathcal{O})$ is irreducible relative to \mathcal{O} , if the corresponding free factor system $\mathcal{G} = \{[G_i] : 1 \leq i \leq q\}$ is a maximal proper, ϕ -invariant free factor system. Therefore we define the notion of an irreducible with irreducible powers (or simply IWIP) automorphism relative to \mathcal{O} , as in the special case where G is a finitely generated free group.

In this paper, we study IWIP automorphisms and in particular we show that we can define the stable (and unstable) lamination Λ associated to an IWIP, using exactly the same method as in the free case. In the classical case, it can be proved that the stabiliser of the lamination is virtually cyclic (see [1]). However, in the general case, the presence of the factor automorphisms of the G_i 's, does not allow us to get the same statement and as we will see this is not true in general, but can prove the following generalisation:

Theorem 1.1. Let ϕ be an IWIP relative to some relative outer space \mathcal{O} . Let's denote by $Stab(\Lambda_\phi) = Stab(\Lambda)$ the $Out(G, \mathcal{O})$ stabiliser of the stable lamination Λ .

- (i) There is a normal periodic subgroup A of $Stab(\Lambda)$, such that the group $Stab(\Lambda)/A$ has a normal subgroup B isomorphic to subgroup of $\bigoplus_{i=1}^q Out(G_i)$ and $(Stab(\Lambda)/A)/B$ is isomorphic to \mathbb{Z} .
- (ii) Let's suppose that $Out(G)$ is virtually torsion free. Then $Stab(\Lambda)$ has a (torsion free) finite index subgroup K such that K/B' is

isomorphic to \mathbb{Z} , where B' is a normal subgroup of K isomorphic to subgroup of $\bigoplus_{i=1}^q \text{Out}(G_i)$.

- (iii) Finally, if we further suppose that every $\text{Out}(G_i)$ is finite, then $\text{Stab}(\Lambda)$ is virtually (infinite) cyclic.

We can find the notion of laminations for a free group, in a lot of different forms and contexts in the literature and study of them implies important results (for example, see [1] [2], [3], [6], [7] and [12]), therefore it looks like interesting to generalise this notion in a more general context. In addition, further motivation is that we can find natural generalisations for a lot of facts about CV_n in the general case, for example in [10], Francaviglia and Martino generalised a lot of tools like train track maps and the Lipschitz metric. But there are some recent papers that show we can also use further methods of studying $\text{Out}(Fn)$ for $\text{Out}(G)$ (where G is written as free product as above) such that the closure of outer space, the Tits alternatives for $\text{Out}(G)$, the hyperbolic complex corresponding to $\text{Out}(G)$ and the asymmetry of the outer space of a free product (see [14], [15], [16], [19] and [24]). Finally, the author as an application of the results of the present paper generalises a result of Hilion ([13]), about the stabiliser of attractive fixed point of an IWIP automorphism ([25]).

Given a group G and an element $g \in G$, a natural question is to study the centraliser $C(g)$ of g in G . In several classes of groups, centralisers of elements are reasonably well-understood and sometimes they are useful to the study of the group. For example, Feighn and Handel in [9] classified abelian subgroups in $\text{Out}(F_n)$ by studying centralisers of elements. Moreover, a well known result for an IWIP automorphism of a free groups (there are several proofs, see [1], [20] or [18]) states that their centralisers are virtually cyclic. Again, it's not true for a relative IWIP, but in the general case, we can obtain a generalisation of this result where the group of factor automorphisms is still appeared and namely:

Theorem 1.2. Let ϕ be an IWIP as above. Let's denote by $C(\phi)$ the centraliser of ϕ in $\text{Out}(G, \mathcal{O})$.

- (i) There is a normal periodic subgroup A_1 of $C(\phi)$, such that the

group $C(\phi)/A_1$ has a normal subgroup B_1 isomorphic to subgroup of $\bigoplus_{i=1}^q Out(G_i)$ and $(C(\phi)/A_1)/B_1$ is isomorphic to \mathbb{Z} .

- (ii) Let's also suppose that $Out(G)$ is virtually torsion free. Then $C(\phi)$ has a (torsion free) finite index subgroup A'_1 such that A'_1/B_1 is isomorphic to \mathbb{Z} , where B_1 is a normal subgroup of A_1 isomorphic to subgroup of $\bigoplus_{i=1}^q Out(G_i)$.
- (iii) Finally, if we further suppose that every $Out(G_i)$ is finite, then $C(\phi)$ is virtually (infinite) cyclic.

In fact, we can get a stronger result for commensurators instead of centralisers.

Note that there are a lot of IWIP automorphisms that don't commute with the factor automorphisms of G_i 's and in particular for them, the theorem above implies that their centralisers are virtually cyclic.

On the other hand, there are examples of IWIP automorphisms which have big centraliser (in particular, they are not virtually cyclic).

Example 1.3. *We fix the free product decomposition $G = G_1 * \langle b_1 \rangle * \langle b_2 \rangle$, where b_i are of infinite order and we denote by $F_2 = \langle b_1 \rangle * \langle b_2 \rangle$ the "free part". Then in the corresponding outer space $\mathcal{O}(F_2, G_1, F_2)$, which we denote by \mathcal{O} . In each tree $T \in \mathcal{O}$ there is exactly one non free vertex v_1 s.t $G_{v_1} = G_1$. Then we define the outer automorphism ϕ , which satisfies $\phi(a) = a$ for every $a \in G_1$, $\phi(b_1) = b_2 g_1$, $\phi(b_2) = b_1 b_2$ for some $g_1 \in G_1$, then we can see that $\phi \in Out(G, \mathcal{O})$ is an IWIP relative to \mathcal{O} . But then every factor automorphism of G_1 that fixes g_1 commutes with ϕ and therefore $C(\phi)$ contains the subgroup A of $Aut(G_1)Inn(G)$ that fixes g_1 . So if A is sufficiently big, the relative centraliser is not virtually cyclic, but the quotient modulo its intersection with A is (by applying the previous theorem). Since we can change G_1 with any group (of finite Kurosh rank) and we can get automorphisms with arbitrarily big centralisers. For example, if G_1 is isomorphic to F_3 and g_1 an element of its free basis, we have that $C(\phi)$ contains a subgroup which is isomorphic to $Aut(F_2)Inn(G)$.*

Strategy of the proof: The paper is organized as follows:

In Section 2, we recall some preliminary definitions, facts and well known results about the outer space of a free product. In Section 3, we prove a useful technical lemma for \mathcal{O} -maps, more specifically we prove that every two such maps are equal except possibly two bounded (depends only on the map, not the path) paths near the endpoints. The next sections form the main part of this paper and we follow exactly the same approach as in [1]. In section 4, we define the lamination using train track representatives, and then we extend the notion to any tree. Also, we list some useful properties. In Section 5, we define the action of $Out(G, \mathcal{O})$ on the set of irreducible laminations. In Section 6 we define the notion of a subgroup which carries the lamination and then we prove that any such subgroup has finite index in the whole group. In Section 7, which is the most crucial for our arguments, we construct a homomorphism from the stabiliser to group of positive real numbers. Then in Section 8, we study the kernel of this homomorphism, and in particular, we prove that any element of the kernel is non-exponentially growing and in the reducible case it has a relative train track representative with a very good form restricted to the lower strata. Also, we prove the discreteness of the image which allows us to think the previous map, as a homomorphism from the stabiliser to the group of integers. Finally, in Section 9, we prove some useful lemmas and the main results.

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2 Preliminaries

2.1 Outer space and \mathcal{O} -maps

In this subsection we recall the definitions of outer space and some basic properties. For example, the existence of \mathcal{O} - maps between any

two elements of the space which is a very useful tool.

Everything in the present and the next subsection about the outer space, the \mathcal{O} - maps and the train track representatives can be found in [10].

Let G be a group which splits as a finite free product of the following form $G = H_1 * \dots * H_q * F_r$, where every H_i is non-trivial, not isomorphic to \mathbb{Z} and freely indecomposable. We say that such a group has *finite Kurosh rank* and such a decomposition is called *Grushko decomposition*. For example, every f.g. group admits a splitting as above (by the Grushko's theorem). We are interested only for groups which have finite Kurosh rank.

Now for a group G , as above, we fix an arbitrary (non-trivial) free product decomposition $G = H_1 * \dots * H_q * F_r$ (without the assumption that the H_i 's are not isomorphic to \mathbb{Z} or freely indecomposable), but we additionally suppose that $r > 0$. These groups admit co-compact actions on \mathbb{R} -trees (and vice-versa). It is useful that we can also apply the theory in the case that G is free, and the G_i 's are certain free factors of G (relative free case).

We will define an outer space $\mathcal{O} = \mathcal{O}(G, (G_i)_{i=1}^p, F_r)$ relative to the free product decomposition (or relative outer space). The elements of the outer space can be thought as simplicial metric G -trees, up to G -equivariant homothety. Moreover, we require that these trees also satisfy the following:

- The action of G on T is minimal.
- The edge stabilisers are trivial.
- There are finitely many orbits of vertices with non-trivial stabiliser, more precisely for every H_i , $i = 1, \dots, q$ (as above) there is exactly one vertex v_i with stabiliser H_i (all the vertices in the orbits of v_i 's are called *non-free vertices*).
- All other vertices have trivial stabiliser (and we call them *free vertices*).

The quotient G/T is a finite graph of groups. We could also define the outer space as the space of "marked metric graph of groups" using the quotients instead of the trees, but we won't use this point of view because here it is easier to work using the trees. However, we use the

quotients when the statements in this context are less complicated. We would like to define a natural action of $Out(G)$ on \mathcal{O} , but this is not possible since it is not always the case that the automorphisms preserve the structure of the trees (i.e. they don't send non-free vertices to non-free vertices). However, we can describe here the action of a specific subgroup of $Out(G)$ (namely, the automorphisms that preserve the decomposition or equivalently the structure of the trees) on \mathcal{O} .

Let $Aut(G, \mathcal{O})$ be the subgroup of $Aut(G)$ that preserve the set of conjugacy classes of the G_i 's. Equivalently, $\phi \in Aut(G)$ belongs to $Aut(G, \mathcal{O})$ iff $\phi(G_i)$ is conjugate to one of the G_j 's. The group $Aut(G, \mathcal{O})$ admits a natural action on a simplicial tree by "changing the action", i.e. for $\phi \in Aut(G, \mathcal{O})$ and $T \in \mathcal{O}$, we define $\phi(T)$ to be the element with the same underlying tree with T , the same metric but the action is given by $g * x = \phi(g)x$ (where the action in the right hand side is the action of the G -tree T). Now since the set of inner automorphisms of G , $Inn(G)$ acts trivially on \mathcal{O} we can define $Out(G, \mathcal{O}) = Aut(G, \mathcal{O})/Inn(G)$ which acts on \mathcal{O} as above. Note that in the case of the Grushko decomposition we have $Out(G) = Out(G, \mathcal{O})$.

We say that a map between trees $A, B \in \mathcal{O}$, $f : A \rightarrow B$ is an \mathcal{O} -map, if it is a G -equivariant, Lipschitz continuous, surjective function. Note here that we denote by $Lip(f)$ the Lipschitz constant of f .

It is very useful to know that there are such maps between any two trees. This is true and, additionally, by their construction they coincide on the non-free vertices (and in section 3, we prove that every two such maps "almost" coincide). More specifically, by [10], we get:

Lemma 2.1. For every pair $A, B \in \mathcal{O}$; there exists a \mathcal{O} -map $f : A \rightarrow B$. Moreover, any two \mathcal{O} -maps from A to B coincide on the non-free vertices.

Let $f : A \rightarrow A$ be a simplicial (sending vertices to vertices and edges to edge-paths) \mathcal{O} -map, where $A \in \mathcal{O}$. Then f induces a map (here we denote by Df the map which sends every edge e to the first edge of the edge path $f(e)$) on the set of turns, sending every turn (e_1, e_2) to the turn $(Df(e_1), Df(e_2))$. Then as usually, we say that the turn (e_1, e_2) is *legal*, if for every k the turn $(Df^k(e_1), Df^k(e_2))$ is non-degenerate. This induces a pre-train track structure on the set

of edges at each vertex. But there are also different pre-train track structures and one of which we will use later, therefore we need the general definition.

- Definition 2.2.** (i) A **pre-train track structure** on a G -tree T is a G -invariant equivalence relation on the set of germs of edges at each vertex of T . Equivalence classes of germs are called **gates**.
- (ii) A **train track structure** on a G -tree T is a pre-train track structure with at least two gates at every vertex.
- (iii) A **turn** is a pair of germs of edges emanating from the same vertex. A **legal turn** is called a turn for which the two germs belong to different equivalent classes. A **legal path**, is a path that contains only legal turns.

A pre-train track structure induced by some \mathcal{O} - map is not always a train track structure, but there are some \mathcal{O} - maps (we call them optimal maps) which induce train track structures. But firstly we need the notion of PL maps (which corresponds to piecewise linear homotopy equivalence in the free case). We call a map between two elements of the outer space **PL**, if it is piecewise linear and \mathcal{O} -map. We denote by $A_{max}(f)$ the subgraph of A consisting on those edges e of A for which $S_{f,e} = Lip(f)$ (i.e. the set of edges which are maximally stretched by f). Note that A_{max} is G -invariant and that in literature the set A_{max} is often referred to as *tension graph*.

As we have seen in the discussion above, for every map there is an induced structure. More specifically, if $A, B \in \mathcal{O}$ and $f : A \rightarrow B$ is a PL-map, then **the pre-train track structure induced by f** on A is defined by declaring germs of edges to be equivalent if they have the same *non-degenerate* f -image (so if two maps that are collapsed by f , they are not equivalent).

We are now in position to define optimal maps:

Definition 2.3. Let $A, B \in \mathcal{O}$. A PL-map $f : A \rightarrow B$ is not optimal at v , if A_{max} has only one gate at v for the pre-train track structure induced by f . Otherwise, f is **optimal at v** . The map f is **optimal**, if it is optimal at all vertices.

Remark. A PL-map $f : A \rightarrow B$ is optimal if and only if the pre-train track structure induced by f is a train track structure on A_{max} . In

particular, if $f : A \rightarrow B$ is an optimal map, then at every vertex v of A_{max} there is a legal turn in A_{max} .

Note also that by [10], every PL-map is optimal at non-free vertices and for every $A, B \in \mathcal{O}$ there exists an optimal map from A to B . Therefore we can always choose our \mathcal{O} - maps to be optimal and we will use optimal maps without further mention.

2.2 Relative Automorphisms

We denote by $Out(G, \{G_i\}^t)$ the subgroup of $Out(G, \mathcal{O})$ made of those automorphisms that act as a conjugation by an element of G on each G_i . Since the G_i 's are free factors of G , each subgroup G_i is equal to its normalizer in G . Therefore, any element of $Out(G, \mathcal{O})$ (i.e. that preserves the conjugacy class of the G_i 's) induces a well-defined outer automorphism of G_i . Therefore there is a natural homomorphism $Out(G, \{G_i\}^t) \rightarrow Out(G_i)$ and by taking the product over all groups G_i , we get a (surjective) homomorphism $Out(G, \mathcal{O}) \rightarrow \bigoplus_{i=1}^p Out(G_i)$, with kernel exactly $Out(G, \{G_i\}^t)$.

2.3 Train Track Maps and Irreducibility

In this section we will define the notion of a "good" representative of an outer automorphism. It is a generalisation of train track representatives of automorphisms of free groups, but as we have already mentioned we work in the trees instead of their quotients. For more details for this approach see [10, 23]. As we have seen there are representatives of every outer automorphism (i.e. \mathcal{O} -maps from A to $\phi(A)$), but sometimes we can find representatives with better properties. These maps, which are called *train track maps*, are very useful and every irreducible automorphism has such a representative (we can choose it to be simplicial, as well).

For $T \in \mathcal{O}$ we say that a Lipschitz surjective map $f : T \rightarrow T$ **represents** ϕ if for any $g \in G$ and $t \in T$ we have $f(gt) = \phi(g)(f(t))$. (In other words, if it is an \mathcal{O} -map from T to $\phi(T)$.) We give below the definition of a train track map representing an outer automorphism. We are interested for these maps because we can control their cancellation (it is not possible to avoid it).

Definition 2.4. If $T \in \mathcal{O}$ then a PL-map $f : T \rightarrow T$, which representing ϕ , is a train track map if there is a train track structure on T so that

- (i) f maps edges to legal paths (in particular, f does not collapse edges)
- (ii) If $f(v)$ is a vertex, then f maps inequivalent germs at v to inequivalent germs at $f(v)$.

In the free case, an automorphism ϕ is called *irreducible*, if there is no ϕ -invariant free factor up to conjugation (or equivalently the topological representatives of ϕ haven't non-trivial proper invariant subgraphs). In our case we know that the G_i 's are invariant free factors, but we don't want to have "more invariant free factors". More precisely, we will define the irreducibility of some automorphism *relative* to the space \mathcal{O} or to the free product decomposition.

Definition 2.5. We say $\Phi \in \text{Out}(G, \mathcal{O})$ is \mathcal{O} -*irreducible* (or simply irreducible) if for any $T \in \mathcal{O}$ and for any $f : T \rightarrow T$ representing Φ , if $W \subseteq T$ is a proper f -invariant G -subgraph then G/W is a union of trees each of which contains at most one non-free vertex.

We can also give an alternative algebraic definition, but we need the notion of a free factor system. Suppose that G can be written as a free product, $G = G_1 * G_2 * \dots * G_p * G_\infty$. Then we say that the set $\mathcal{A} = \{[G_i] : 1 \leq i \leq p\}$ is a **free factor system** for G , where $[A] = \{gAg^{-1} : g \in G\}$ is the set of conjugates of A .

Now we define an order on the set of free factor systems for G . More specifically, given two free factor systems $\mathcal{G} = \{[G_i] : 1 \leq i \leq p\}$ and $\mathcal{H} = \{[H_j] : 1 \leq j \leq m\}$, we write $\mathcal{G} \sqsubseteq \mathcal{H}$ if for each i there exists a j such that $G_i \leq gH_jg^{-1}$ for some $g \in G$. The inclusion is strict, and we write $\mathcal{G} \subset \mathcal{H}$, if some G_i is contained strictly in some conjugate of H_j . We can see $\{[G]\}$ as a free factor system and in fact, it is the maximal (under \sqsubseteq) free factor system. Any free factor system that is contained strictly to \mathcal{G} is called **proper**. Note also that the Grushko decomposition induces a free factor system, which is actually the minimal free factor system (relative to \sqsubseteq). A more detailed discussion for the theory of free factor systems can be found in [12].

We say that $\mathcal{G} = \{[G_i] : 1 \leq i \leq p\}$ is ϕ - **invariant** for some $\phi \in \text{Out}(G)$, if ϕ preserves the conjugacy classes of G_i 's. We are only interested for free factor systems that G_∞ is a finitely generated free group. In particular, we suppose that $G = G_1 * G_2 * \dots * G_p * G_\infty$, and $G_\infty = F_k$ for some f.g. free group F_k . In each free factor system $\mathcal{G} = \{[G_i] : 1 \leq i \leq k\}$, we associate the outer space $\mathcal{O} = \mathcal{O}(G, (G_i)_{i=1}^p, F_k)$ and any $\phi \in \text{Out}(G)$ leaving \mathcal{G} invariant, will act on \mathcal{O} in the same way as we have described earlier.

Definition 2.6. Let \mathcal{G} be a free factor system of G which is Φ -invariant for some $\Phi \in \text{Out}(G)$. Then Φ is called *irreducible relative to \mathcal{G}* , if \mathcal{G} is a maximal (under \sqsubseteq) proper, Φ -invariant free factor system.

The next lemma confirms that the two definitions of irreducibility are related.

Lemma 2.7. Suppose \mathcal{G} is a free factor system of G with associated space of trees \mathcal{O} , and further suppose that \mathcal{G} is ϕ -invariant. Then ϕ is irreducible relative to \mathcal{G} if and only if ϕ is \mathcal{O} -irreducible.

Moreover, one interesting fact is that for an irreducible automorphism we can give a characterisation of train track maps using the axes of hyperbolic elements. More specifically, if ϕ is irreducible, then for a map f representing $\phi \in \text{Out}(G, \mathcal{O})$, to be a train track map is equivalent to the condition that there is $g \in G$ (hyperbolic element) so that $L = \text{axis}_T(g)$ (the axis of g) is legal and $f^k(L)$ is legal $k \in \mathbb{N}$.

Now let's give the definition of an irreducible automorphism with irreducible powers relative to \mathcal{O} , which are the automorphisms that we will study.

Definition 2.8. An outer automorphism $\phi \in \text{Out}(G, \mathcal{O})$ is called **IWIP** (*irreducible with irreducible powers* or *fully irreducible*), if every ϕ^k is irreducible relative to \mathcal{O} .

The next theorem is very important since we can always choose representatives of irreducible automorphisms with nice properties, as in the free case. It generalises the well known theorem of Bestvina and Handel (see [4]). In particular, we can apply it on every power of some *IWIP*.

Theorem 2.9 (Francaviglia- Martino). Let $\phi \in \text{Out}(G, \mathcal{O})$ be irreducible. Then there exists a (simplicial) train track map representing ϕ .

The discussion above implies that we can always find an optimal train track representative of an irreducible $\phi \in \text{Out}(G, \mathcal{O})$. This map has the property that the image of every legal path (in particular, of edges) is stretched by a constant number $\lambda \geq 1$ which depends only on ϕ .

We close this subsection with an interesting remark.

Remark. Every outer automorphism $\phi \in \text{Out}(G)$ is irreducible relative to some appropriate space (or relative to some free product decomposition).

2.4 Bounded Cancellation Lemma

Let $T, T' \in \mathcal{O}$ and $f : T \rightarrow T'$ be an \mathcal{O} -map. If we have a concatenation of legal paths ab where the corresponding turn is illegal, then it is possible to have cancellation in $f(a)f(b)$. But the cancellation is bounded, with some bound that depends only on f and not on a, b . In particular, we can define the bounded cancellation constant of f (let's denote it $BCC(f)$) to be the supremum of all real numbers N with the property that there exist A, B, C some points of T with B in the (unique) reduced path between A and C such that $d_T(f(B), [f(A), f(C)]) = N$ (the distance of $f(B)$ from the reduced path connecting $f(A)$ and $f(C)$), or equivalently is the lowest upper bound of the cancellation for a fixed \mathcal{O} -map.

The existence of such number is well known, for example a bound has given in [14]:

Lemma 2.10. Let $T \in \mathcal{O}$, let $T' \in \mathcal{O}$, and let $f : T \rightarrow T'$ be a Lipschitz map. Then $BCC(f) \leq \text{Lip}(f) \text{qvol}(T)$, where $\text{qvol}(T)$ the quotient volume of T , defined as the infimal volume of a finite subtree of T whose G -translates cover T .

We can also, exactly as in the free case, define a critical constant, C_{crit} corresponding to a train track map. Let's suppose that f is train track map with expanding factor λ (for

example, a train track representative of some IWIP ϕ). If we take a, b, c legal paths and abc is a path in the tree, and let's denote $l = \text{length}(b)$ the length of the middle segment. If we suppose further that satisfies $\lambda l - 2BCC(f) > l$, then iteration and tightening of abc will produce paths with the length of the legal leaf segment corresponding to b to be arbitrarily long. This is equivalent to require that $l > \frac{2BCC(f)}{\lambda-1}$, and we call the number $C_{crit} = \frac{2BCC(f)}{\lambda-1}$, the *critical constant* for f . For every C that exceeds the critical constant there is $m > 0$ such that b , as above, has length at least C then the length of the legal leaf segment of $[f^k(abc)]$ corresponding to b is at least $m\lambda^k \text{length}(b)$. Therefore we can see that any path which contains a legal segment of length at least C_{crit} , has the property that the lengths of reduced f -iterates of the path are going to infinity.

2.5 N-periodic paths

A difference between the free and the general case is that it is not always the case that there are finitely many orbits of paths of a specific length (if there are non-free vertices with infinite stabiliser), but it is true that there are finitely many paths that have different projection in the quotient. Therefore the role of Nielsen periodic paths play the N-periodic paths that we define below. Note that if $h : S \rightarrow S$, we say that a point $x \in S$ is h -periodic, if there are $g \in G$ and some natural k s.t. $h^k(x) = gx$.

Definition 2.11. (i) Two paths p, q in $S \in \mathcal{O}$ are called *equivalent*, if they project to the same path in the quotient G/S . In particular, their endpoints $o(p), o(q)$ and $t(p), t(q)$ are in the same orbits, respectively.

(ii) Let $h : S \rightarrow S$ be a representative of some outer automorphism ψ , let p be a path in S and let's suppose that the endpoints of p are h -periodic (with period k), then we say that a path p in S is *N-periodic* (with period k), if the paths $[h^k(p)], p$ are equivalent.

Geometric and non-Geometric automorphisms: We will define here some notions for automorphisms that have been motivated by the properties of geometric and non-geometric automorphisms, respectively. The terminology also comes from the free case. In that

case, we say that ϕ is geometric if it can be represented as a (pseudo-Anosov) homeomorphism of a punctured surface. It is well known that for the non-geometric case there is an integer m such that it is impossible to concatenate more than m indivisible Nielsen paths for every map f which represents ϕ . We will generalise this property in order to give our definitions, using the notion of an indivisible N -periodic path as in the free case. In particular:

Definition 2.12. We say that some ϕ has the *NGC* property, if it is impossible to concatenate more than m indivisible N -periodic paths for every \mathcal{O} -map f which represents ϕ . Otherwise, we say that ϕ has the *GC* property.

2.6 Relative train-track maps

Having good representatives of outer automorphisms, is very useful. If our automorphism is irreducible, it is possible to find train track representatives, as we have seen. But even in the reducible case we can find relative train track representative. The existence of such maps it follows from [10] or [5].

That we have is that every automorphism can be represented as an \mathcal{O} -map $f : T \rightarrow T$ such that T has a filtration $T_0 \subseteq T_1 \subseteq \dots \subseteq T_k = T$ by f -invariant G -subgraphs, where T_0 contains every non-free vertex, we denote by $H_r = cl(T_r - T_{r-1})$ and we suppose that the transition matrix (it can be defined as in the free case but we count orbits of edges) of every H_r is irreducible (or zero matrix) so we can correspond in every H_r some PF eigenvalue (let's denote it λ_r). In addition, f has some train track properties (such as mixed turns are legal and the map is r -legal). There is a very interesting corollary that we will use: for every edge-path a in H_r , the reduced image of a , $[f(a)]$, can be written as a concatenation of non-degenerate edge-paths in T_{i-1} and H_i with the first and the last contained in H_i .

For such a , we can distinguish between two cases for the strata: if there exists some edge of e in H_r such that $[f(e)]$ contains at least two copies (orbits) of e , then we say that the stratum is *exponentially growing* and we can see the r -lengths of images of edges in H_r expands by $\lambda_r > 1$ and in particular the lengths of reduced f -iterates of edges in H_r are going to infinity (using the train track properties). Otherwise, the stratum called non-exponentially growing and the map f (if we ignore

the lower strata) is just a permutation of edges of the same length. An automorphism is called *exponentially growing* if some representative has at least one exponentially growing stratum. In other case, it is called *non-exponentially growing* automorphism.

2.7 Graph of Groups and Subgroups

We will recall only some facts for the graph of groups. For more about graph of groups and their subgroups, see [22].

In the special case that we are interested, a graph of groups can be defined as a finite connected graph X (let call Γ the underlying graph) for which in every vertex v we correspond some (vertex) group G_v . We call *non-free* the vertices for which the corresponding group is non-trivial. Then the fundamental group of X , $\pi_1(X)$ is the free product of $\pi_1(\Gamma)$ (which is a f.g. free group) and the vertex groups.

We will use a specific kind of subgroups of $\pi_1(X)$. Let γ be a loop in $v_0 \in V(\Gamma)$. Then starting from v_0 and following the path of γ we meet some non-free vertices (we can return back also, but we have always follow γ). So we can read words of a fixed form, and this process produces words of the fundamental group (we can see it as the group which it consists of all the words constructed as above but without fixing some loop γ). In fact, the set of all such words is a subgroup of $\pi_1(X)$, which corresponds to γ .

3 Every two \mathcal{O} - maps coincide

In [10] it has been proved the existence of \mathcal{O} -maps. We will prove that even if in the construction of such maps there is a lot of freedom, the reduced images of all of them coincide, up to bounded error. As a consequence we obtain that their lengths are comparable.

Theorem 3.1. Let $f, h : A \rightarrow B$ be \mathcal{O} - maps. Then there exists a positive constant C (which depends only on f , h and A), so that for every path L in A , then $[f(L)]$ and $[h(L)]$ are equal, except possibly some subpaths near their endpoints which their lengths are bounded by C .

Proof. Firstly, we suppose that there is at least one non-free vertex which we denote it by v . Then we have that $f(v) = h(v)$. If $L = [a, b]$

is an edge - path, then in distance at most $vol(A/G)$, we can find vertices of the form g_1v, g_2v near a, b respectively such that $[a, b] \subseteq [g_1v, g_2v]$. Then $[f(L)]$ is contained in $[f(g_1v), f(g_2v)]$, except possibly some segments near a, b of length at most $C' = vol(A/G)Lip(f)$. Similarly, we apply the same argument for $[h(g_1v), h(g_2v)]$ and we get a constant $C'' = vol(A/G)Lip(h)$. Therefore since $[h(g_1v), h(g_2v)] = [f(g_1v), f(g_2v)]$, we get $[f(L)] = [h(L)]$ except possibly some segments near a, b which are bounded by $C = \max(C', C'')$ (by definition depends only on $Lip(f), Lip(h), vol(G/A)$)

If there are no non-free vertices, we are in the free case and the result is well known. \square

Note also that it is not difficult to see that every \mathcal{O} -map is a quasi-isometry.

4 Laminations

We follow exactly the same approach as in [1] and some of the proofs are essentially the same, but since in this context the definitions have adjusted appropriately, we give detailed proofs for the convenience of the reader. On the other hand, there are a lot of technical issues which are not appeared in the free case and they are addressed separately. In this section we define the notion of the lamination associated to an IWIP. Firstly, we use the train track maps to define the lamination in a specific tree and the existence of \mathcal{O} -maps between any two trees allows us to generalise it for every tree.

4.1 Construction of the lamination and properties

Let $\phi \in Out(G, \mathcal{O})$ be an (expanding) irreducible automorphism, with irreducible powers and $f : A \rightarrow A$ for some $A \in \mathcal{O}$ be a train track map which represents ϕ (so it satisfies $f(gx) = \phi(g)f(x)$). We can also suppose that f expand the length of the edges by a uniform factor $\lambda > 1$ (this can be done if we choose an optimal train track that represents f , as we have already seen).

By changing f with some iterate, if necessary, we can suppose that there is $x \in A$ which is a periodic point ($f^k(x) = x$, for some k), in the interior of some edge (in general there exists x s.t. $f^k(x) = gx$

since the quotient is finite, but we can change the space A , changing isometrically the action, with $\phi_g(A)$ and there the requested property holds). Now let U some ϵ -neighbourhood, for some small ϵ (we want the neighbourhood to be contained in the interior of the edge) and then there is some $N > 0$ s.t. $f^N(U) \supset U$.

We can choose an isometry $\ell : (-\epsilon, \epsilon) \rightarrow U$ and extend it to the unique isometry $\ell : \mathbb{R} \rightarrow A$ s.t. $\ell(\lambda^N t) = f^N(\ell(t))$ and then we say that the bi-infinite line ℓ is *obtained by iterating a neighbourhood of x* .

Definition 4.1. • We say that two isometric immersions $A : [a, b] \rightarrow A$ and $B : [c, d] \rightarrow A$, where $a, b, c, d \in \mathcal{R}$ are equivalent, if there exists an isometry $q : [a, b] \rightarrow [c, d]$ s.t the triangle commutes ($Bq = A$). (This relation is an equivalence relation on the set of isometric immersions from a finite interval to A).

- If P is an equivalence class and we choose a representative of that class $\gamma : [a, b] \rightarrow A$, we can define $f(P)$ as the equivalence class of $f\gamma : [a, b] \rightarrow A$, pulled tight and scaled so it is an isometric immersion.
- A leaf segment of an isometric immersion $\mathbb{R} \rightarrow A$ is the equivalence class of the restriction to a finite interval.

Let ℓ be an isometric immersion, then we correspond the G -set I_ℓ (of the leaf segments of ℓ) to ℓ . We can also define an equivalence relation on the set of isometric immersions from \mathbb{R} to A .

Definition 4.2. Let ℓ, ℓ' be two isometric immersions from \mathbb{R} to A , then we say that they are equivalent if $I_\ell = GI_{\ell'}$. Namely, we say that they are *equivalent* if for every leaf segment P of ℓ there is an element $g \in G$ and Q a leaf segment of ℓ' s.t. $P = gQ$ and vice versa (or equivalently every l.s. of ℓ is mapped by some g to a l.s. of ℓ')

Remark. Here note that it is obvious that if $\ell(t) = g\ell'(t)$ (ℓ, ℓ' are in the same orbit), then ℓ and ℓ' are equivalent.

We will prove that if we construct any other line by iterating a neighbourhood of any other periodic point (here we mean that there is k and $g \in G$ s.t. $f^k(x) = gx$) then it is equivalent with ℓ .

Lemma 4.3. Let $y \in A$, be any other f -periodic point in the interior of some edge of A and ℓ' is the obtained by iterating of some neighborhood of y . Then ℓ and ℓ' are equivalent.

Proof. We will show that any l.s. of ℓ is mapped by some element of G to a l.s. of ℓ' , then the converse follows by symmetry.

Since f represents an irreducible automorphism (and the same holds for every power of f), ℓ' contains some orbit of every edge, so in particular if x is contained in the interior of the edge e we have that there exists some $g \in G$, s.t. $gx \in ge \subseteq \ell'$. So there is an isometry $\psi : (-\epsilon, \epsilon) \rightarrow (a - \epsilon, a + \epsilon)$ with the property $\ell(t) = g\ell'(\psi(t))$.

Let N' be a natural number s.t. $\ell'(\lambda^{N'}t) = f^{N'}(\ell(t))$ and then for any $t \in U$ (U as in the definition) we have that $\ell(\lambda^{kNN'}t) = f^{kNN'}(\ell(t)) = f^{kNN'}(g\ell'(\psi(t))) = \phi^{kNN'}(g)f^{kNN'}(\ell'(\psi(t))) = \phi^{kNN'}(g)\ell'(\lambda^{kNN'}\psi(t))$. But since every prechosen interval is contained in some interval of the form $\lambda^{kNN'}(-\epsilon, \epsilon)$ for large k , we have that for every l.s. of ℓ is mapped by some $\phi^{kNN'}(g) \in G$ to some l.s. of ℓ' . \square

We are now in position to define the stable lamination corresponding to A .

Now the **stable lamination** in A -coordinates $\Lambda = \Lambda_f^+(A)$ is the equivalence class of isometric immersions from \mathbb{R} to A containing some (and by previous lemma any) immersion obtained as above (by iterating a neighborhood of a periodic point). We call the immersions representing Λ **leaves** of Λ and the leaf segments (l.s.) of some leaf of Λ **leaf segments** of Λ (by definition of the equivalence relation, every leaf of Λ contains some orbit of every l.s. of Λ).

Note that the every leaf of the lamination project to the same bi-infinite path in the quotient.

We will list some useful properties of the stable lamination.

Proposition 4.4. (i) Any edge of A is a leaf segment of Λ .

(ii) Any f -iterate of a leaf segment is a leaf segment.

(iii) Any subsegment of a leaf segment is a leaf segment.

(iv) Any leaf segment is a subsegment of a sufficiently high iterate of an edge.

- (v) For any leaf segment P there is a leaf segment P' such that $f(P') = P$.
- (vi) Let a be a segment which is the period of the axis of some hyperbolic element which crosses k edges (counted with multiplicity). Then any f -iterate of a (pulled tight) can be written as concatenation of less or equal k leaf segments.

Proof. (i) This is clear by the proof of the previous lemma, since f represents an irreducible automorphism and this implies that every ℓ contains orbits of every edge, so if ge is contained in ℓ then e is contained in $g^{-1}\ell$ which is equivalent to ℓ thus is a leaf of Λ , and as consequence e is leaf segment of a leaf therefore it is l.s. of Λ .

(ii) Firstly, we note that if x is f -periodic then $f(x)$ is f -periodic with the same period (in fact every $f^m(x)$ is periodic) and let's denote ℓ' the isometric immersion constructed as above, so if P is a l.s. of ℓ , then $f(P)$ is a l.s. of ℓ' but since ℓ, ℓ' are equivalent by lemma, we have that ℓ' is a leaf of Λ and therefore $f(P)$ is a l.s. of Λ . So we can do it for every iterate of f .

(iii) This is obvious, since we restrict the isometric immersion to the subsegment and it is a l.s. of a leaf of Λ and as a consequence a l.s. of Λ .

(iv) We have that f expands the length of every edge by λ , but we can use for representative the isometric immersion constructed as above (by iterating a periodic neighborhood) and the edge in which the periodic point belongs, then by construction of ℓ every l.s. is contained in an high iterate of this edge. For any other representative ℓ' now we can translate ℓ as above (by some element $g \in G$) to have a common segment that contain the prechosen l.s. and the proof reduced to the first case.

(v) Let P be a l.s. of Λ . By (iv) we have that there exists some iterate of an edge and so by ℓ an iterate of a l.s. P'' s.t. P is contained in $f^m(P'')$ and since iterates of l.s. are l.s. and subsegments are l.s. as well, we have that there is P' subsegment of $f^{m-1}(P'')$ with the property $P = f(P')$

(vi) This is obvious since edges are l.s. and f -iterates of l.s. are l.s.. \square

We note that (ii) implies that $f^k(\ell)$ is a leaf of the lamination, for every k .

Definition 4.5. We say that a sequence a_i of isometric immersions $[0, 1]_i \rightarrow A$ (where the metric on $[0, 1]_i$ is scalar multiple of the standard part which depends on i), (weakly) converges to Λ , if for every $L > 0$ the ratio,

$$\frac{m(\{x \in [0, 1]_i | \text{the } L\text{-nbhd of } x \text{ is a leaf segment}\})}{m([0, 1]_i)}$$

converges to 1.

Proposition 4.6. Suppose that a is a segment in A which corresponds to the period of the axis of some hyperbolic element, which is not N-periodic. Then the sequence (of tightenings of $f^i(a)$), $[f^i(a)]$ weakly converges to Λ .

Note that such hyperbolic elements always exist. For example the basis elements of the free group, are not N-periodic by definition of irreducibility.

Proof. Suppose that a can be written as a concatenation of k l.s. then we have $k - 1$ illegal turns (we don't count the endpoints) and since f is a train track map we have that the number of illegal turns in $[f^k(a)]$ is non-increasing so it contains less than or equal to $k - 1$ l.s.. Therefore if the lengths of reduced iterates of a is bounded, and since there are finitely many inequivalent paths with length less than or equal to a specific number, we have that a is N-preperiodic and therefore periodic because a corresponds to a group element, which leads to a contradiction to the hypothesis. Therefore some $[f^i(a)]$ contains arbitrarily long legal segments ($> C_{crit}$), and since the length of $[f^j(a)]$ expands for large j , we have that there are finitely many L -nbds contain points without the requested property (of the endpoints of the concatenation of l.s. so at most k) and the measure of these is at most $2Lk$, as a consequence the ratio converges to 1. \square

Definition 4.7. An isometric immersion $l : \mathbb{R} \rightarrow A$ is quasiperiodic (qp), if for every $L > 0$ there exists $L' > 0$ s.t. for every l.s. P of ℓ of length L and for every l.s. Q of length L' there is $g \in G$ s.t. $gP \subseteq Q$ (P is mapped by g to a subsegment of Q).

Proposition 4.8. Every leaf of Λ is quasiperiodic.

Proof. We will first prove it for some ℓ which has constructed by iterating neighbourhood of a periodic point.

We first verify it for leaf segments Π that consists of only two edges. If we choose $L_0 > 2\max_e(\text{len}(e))$, then if a l.s. P has length $\geq L_0$, then it contains a subleaf segment which is an edge. Then there is N (we can also choose it to be multiple of k) s.t. f^N restricted to any edge crosses some orbit of every turn that they crossed by leaves of $\Lambda_f^+(A)$. So in particular for the chosen Π the iterate of f takes the orbit of that turn, so there exists $g \in G$ such that $\Pi \subseteq gf^N(P)$. Now if P' is any l.s. of length $\lambda^N L_0$, then $P' = f^N(P)$ for some P l.s. of length L_0 and therefore $\Pi \subseteq gP'$.

For the general case, let $L > 0$ be given, then there is $M > 0$ (we choose it to have the property $\lambda^{-M}L < 2\min(\text{len}(e))$) s.t. any l.s. of length $\leq \lambda^{-M}L$ is a subsegment of a two-edge l.s. as above and let $L' = \lambda^{M+N}L_0$.

So let P be a l.s. of length L and P' be a l.s. of length L' . Then by the properties we have that $P = f^M(\Pi)$ where Π is contained to a l.s. as in the special case (by the choice of M , since Π has length $\lambda^{-M}L$), and similarly $P' = f^M(\Pi')$ for a l.s. Π' of length $\lambda^N L_0$. By the special case we have that $\Pi \subseteq g\Pi'$ and this implies that $P = f^M(\Pi) \subseteq \Phi^M(g)f^M(\Pi') = \Phi^M(g)P'$. Since ℓ is $\Phi^M(g)$ -invariant, we have the requested property.

For any other equivalent isometric immersion ℓ' , if we have P l.s. of length L and Q l.s. of length L' then we can find an isometric immersion ℓ like the first case with Q as common segment. Then by the equivalence there exists $g_1 \in G$ s.t. g_1P is l.s. of ℓ , and by quasiperiodicity of ℓ , there is g_2 s.t. $g_2P \subseteq Q$ and g_2P is a l.s. of ℓ' , so we have that ℓ' is quasiperiodic. \square

4.2 Lamination in every tree

Suppose that $f : A \rightarrow A$ and $\Lambda_f^+(A)$ as above and $B \in \mathcal{O}$. Then we know that there exists an optimal map (in particular \mathcal{O} -map) $\tau : A \rightarrow$

B . Then for any immersion $\ell : \mathbb{R} \rightarrow A$ we denote by $\tau(\ell) : \mathbb{R} \rightarrow B$ the unique (up to precomposition by an isometry of \mathbb{R}) pulled tight to be the isometric immersion corresponding to $\tau\ell$.

Lemma 4.9. • If $\ell, \ell' : \mathbb{R} \rightarrow A$ are equivalent leaves, then $\tau(\ell), \tau(\ell')$ are equivalent.

- If ℓ is quasiperiodic, then $\tau(\ell)$ is quasiperiodic.

Proof. Every optimal map τ by [10], can be factored as the composition of a homeomorphism and a finite sequence of folds. We have just to prove that the lemma is true for homeomorphism and folds.

Firstly, let suppose that τ is homeomorphism. In particular $[\tau(\ell)] = \tau(\ell)$ and the same holds for ℓ' as well.

Let P' is a l.s. of $\tau(\ell)$, then there is some l.s. of ℓ P s.t. $P' = \tau(P)$, so there is a translation of P by some element of the group, gP which is contained in ℓ' , therefore $\tau(gP) = gP'$ is contained in $\tau(\ell')$. The converse follows by symmetry and so $\tau(\ell)$ and $\tau(\ell')$ are equivalent.

Suppose now that ℓ is quasiperiodic, fix a $L > 0$ let P' l.s. of $\tau(\ell)$ of length L . Then there is a l.s. P of length at most K (by Bounded Cancellation Lemma there exists such K which doesn't depend on P but only on L) s.t. $P' = \tau(P)$. Then we can define $L'' = L' \text{Lip}(\tau)$, where L' is the constant corresponding by quasiperiodicity to K and we have that if we choose any Q' l.s. of $\tau(\ell)$ of length L'' then there exists a l.s. Q of ℓ of length at least L' s.t. $\tau(Q) = Q'$. Then Q contains orbits of any l.s. of length at most K , in particular it contains some translation of P for some $g \in G$ and therefore as above Q' contains some translation of P' . So $\tau(\ell)$ is quasiperiodic.

We suppose that τ is an equivariant isometric simple fold of some segments starting from the same point v and has the same τ -image, let call them a, b and c be the corresponding segment in the quotient.

For the first statement, we note that is obvious for a l.s. of $[\tau(\ell)]$ which don't contain some orbit of c , since there τ is the identity. On the other hand, if P' is l.s. of $\tau(\ell)$ which contains some orbits of c , then there exists P which contain the same number of orbits as the folded turn and $[\tau(P)] = P'$ (it is concatenation of the segments before and after the folds). Since ℓ, ℓ' are equivalent we have that we can find $g \in G$ s.t. gP is contained in ℓ' , then $[\tau(gP)]$ is a l.s. of $[\tau(\ell')]$. But $[\tau(gP)]$ is just a translation (by g) of $\tau(P)$, and therefore as above we

obtain that $[\tau(\ell)], [\tau(\ell')]$ are equivalent.

For the quasiperiodicity of $[\tau(\ell)]$ we fix a number $L > 0$ and we call M the maximum number of orbits of v which there are in a segment of length L , and L' is the number corresponds by quasiperiodicity for $L'' = L + 2M\text{len}(a)$. Now let P' be a l.s. of length L , then there is P which contains the same number of orbits of the folded turn and $[\tau(P)] = P'$ as above. Then P has length at most L'' , some translation of it is contained in every l.s. of ℓ of length L' . Now let choose Q any l.s. of $[\tau(\ell)]$ of length L' then the preimage has length at least L' , and therefore the preimage has the requested property. So Q contains a translation of P' as above. \square

Definition 4.10. The stable lamination of $f : B \rightarrow B$ in the B -coordinates is the equivalence class $\Lambda_f^+(B)$ containing $\tau(\ell)$ for some (and by previous lemma any) leaf of $\Lambda_f^+(A)$.

Using again the property that τ is factored as the composition of a homeomorphism and a finite sequence of folds combined with the result for the $\Lambda_f^+(A)$, we have the following proposition.

Proposition 4.11. Let a be a segment which is the period of the axis of a hyperbolic element in A , which is not N -periodic. Then the sequence $\{[\tau(f^i(a))]\}$ weakly converges to $\Lambda_f^+(B)$

Lemma 4.12. Suppose that $h : B \rightarrow B$ is any other train track map representing Φ . Then $\Lambda_f^+(B) = \Lambda_h^+(B)$

Proof. Let a be a periodic segment as in 4.6 and 4.11. Then the sequences $[\tau(f^i(a))], [h^i(\tau(a))]$ weakly converge to $\Lambda_h^+(B)$ and to $\Lambda_f^+(B)$, respectively by the previous propositions. But $\tau f^i, h^i \tau$ are \mathcal{O} -maps from A to $\phi(B)$, so their reduced images coincide in every path, after deleting some bounded segments near endpoints. Then there are leaves ℓ, ℓ' of $\Lambda_h^+(B)$ and $\Lambda_f^+(B)$ respectively with arbitrarily long common leaf segments. Since they are both quasiperiodic, it follows that they are equivalent. Indeed, let P be a l.s. of ℓ of length L then there exists L' s.t. for every l.s. of length L' , P' there is some $g \in G$ s.t. $P \subseteq gP'$. But we can find a common segment Q of ℓ and ℓ' of length at least L' , so by quasiperiodicity $P \subseteq gQ \subseteq \ell$ and since $Q \subseteq \ell'$ we have that $P \subseteq gQ \subseteq g\ell'$ and therefore $g^{-1}P \subseteq \ell'$.

We have proved that for every l.s. of ℓ there is an element of the group

that map this l.s. to a l.s. of ℓ' and similarly we can prove the converse so ℓ and ℓ' are equivalent by definition. Therefore $\Lambda_h^+(B) = \Lambda_f^+(B)$ \square

So we have proved that we can use any train track representative to define the set of laminations, in particular we give the following definition:

Definition 4.13. The **stable lamination** Λ_Φ^+ associated to some IWIP $\Phi \in \text{Out}(G, \mathcal{O})$ is the collection $\{\Lambda_f^+(B) | B \in \mathcal{O}\}$ where $f : A \rightarrow A$ is a train track representative of Φ . The **unstable lamination** Λ_Φ^- of Φ is the stable lamination of Φ^{-1} .

5 Action

Let ϕ be an IWIP and $f : T \rightarrow T$ be an optimal train track representative of ϕ .

We denote by \mathcal{IL} the set of stable laminations Λ_ϕ^+ , as ϕ ranges over all IWIP automorphisms relative to \mathcal{O} . The group $\text{Out}(G, \mathcal{O})$ acts on \mathcal{IL} via

$$\psi \Lambda_\phi^+ = \Lambda_{\psi\phi\psi^{-1}}^+ \quad (5.1)$$

More specifically, if ℓ is a leaf of Λ_ϕ^+ in the S -coordinates and $h : S \rightarrow S$ an \mathcal{O} map representing ψ , then $[h(\ell)]$ represents a leaf of $\Lambda_{\psi\phi\psi^{-1}}^+$.

We are interested to study the stabiliser of the action for a fixed automorphism. Note that obviously the centraliser, which we denote by $C(\phi)$, of the IWIP ϕ in $\text{Out}(G, \mathcal{O})$ is a subgroup of $\text{Stab}(\Lambda)$.

We will equip T with a specific train-track structure, the *minimal train-track structure*; more specifically we declare a turn legal, if it is crossed by some leaf of Λ_f^+ . The properties of the lamination imply that a turn is legal iff there is a f -iterate of an edge of T that crosses the turn.

6 Subgroups carrying the lamination

From now and for the rest of the sections, we fix a group of finite Kurosh rank G with some (non-trivial) free product decomposition, the relative outer space \mathcal{O} which corresponds to this decomposition, some expanding IWIP ϕ relative to \mathcal{O} and the associated lamination

$$\Lambda_\phi^+ = \Lambda.$$

This section is devoted to prove that it is not possible for a proper subtree to contain every leaf of the lamination. Moreover, we will prove that every relative train track representative of some automorphism of the stabiliser, after passing to some power, induces the identity on the quotient restricted to any proper invariant subgraph (which is union of strata).

Definition 6.1. Let A be a subgroup of G of finite Kurosh rank, and let's denote $T \in \mathcal{O}$ and T_A the minimal invariant A -subtree. We suppose also that for every $v \in V(T_A)$, $\text{Stab}_A(v) = \text{Stab}_G(v)$. Then we say that A carries the lamination Λ , if there exist some leaf ℓ of Λ which is contained in T_A .

Remark. (i) Every two leaves of the lamination project to the same bi-infinite path in Γ .

(ii) For every vertex v of T there exist $g \in G$ s.t. $gv \in T_A$ (in particular, T_A contains some orbit of any non-free vertex).

Proposition 6.2. If a A is a subgroup of G , as in the previous definition, which carries Λ_ϕ^+ then A has finite index in G .

Proof. Let $f : T \rightarrow T$ be a train-track representative of ϕ , $\Gamma = G/T$ and let $H \rightarrow \Gamma$ be an isometric immersion corresponding to $A \leq G$. Then by our assumptions H is finite graph of groups and by the remarks contains every non-free vertex. Therefore (using also the assumption that the corresponding vertex groups are full), we can complete the immersion, by adding vertices (with trivial vertex group) and edges, to a connected finite-sheeted covering space $p : \Gamma' \rightarrow \Gamma$ and therefore we have that $T' = T$ (where T' is Bass-Serre tree of Γ').

Now we know that if A has infinite index, then we are really adding new edges in Γ' or equivalently we add new orbits of edges in T . But then using irreducibility we can reach a contradiction.

More specifically, we choose e (edge of T) such that $f(e)$ starts with e . Then for every n the path $f^n(e)$ is a path of T_A . So if we choose any edge e_1 (lift of some edge in $\Gamma' - H$) there does not exist n and $g \in G$ such that $f^n(ge)$ passes through e_1 (since e_1 is in different orbit of edges in T_A), but this contradicts the fact that the transition matrix corresponding to f , denote it by $A(f)$, is irreducible. As a consequence, A must have finite index in G . \square

Proposition 6.3. Let $\psi \in \text{Stab}(\Lambda)$, and let $h : S \rightarrow S$ be a relative train-track representative of ψ . Then we can find some S' in \mathcal{O} (which is topologically the same with S , but possibly with different marking) and a relative train track representative $h : S' \rightarrow S'$ of ψ with the following property: let denote by S_0 some h -invariant G -subgraph of S' (without free vertices of valence 1) that is a union of strata. Then there is a k s.t. if we restrict h^k to S_0 induces the identity in G/S_0 .

Proof. Let ℓ be a leaf in S -coordinates and let S_0 be a proper h -invariant subgraph. The quasiperiodicity implies that there is an upper bound to the length of both S_0 and $S - S_0$ segments, and hence only finitely many segments occur (since there are finitely many lengths corresponding to edge-paths of bounded length in the quotient and quasiperiodicity implies that there are finitely many orbits of leaf segments of a specific length). Using the same argument we have that it is not possible for ℓ to contain arbitrarily long segments of a proper subgraph since then the quasiperiodicity implies that ℓ is contained in that subgraph which contradicts to the previous proposition. Therefore ℓ is a concatenation of non-degenerate segments in S_0 and in $S - S_0$ (otherwise would lift to a proper subgraph of H , which is impossible as we have noticed). Now we have that all S_0 -segments are h -preperiodic (there exist M, N s.t. $h^M(L), h^N(L)$ are in the same orbit) or else h -iteration will produce arbitrarily long leaf segments contained in S_0 contradicting quasiperiodicity.

We can start with the disjoint union X of copies of the segments and the natural immersion $X \rightarrow S$ and we identify two endpoints of X if they are mapped to the same point of S . Then fold to convert the resulting map to an immersion $\pi : X' \rightarrow S$. But ℓ lifts to X' (by construction) and so by previous proposition we have again that $X' = S$ (it corresponds to a finite covering space of graph of groups). In particular, any simple periodic segment (the period of the axis of a simple loop or a loop corresponding to an element of some G_{v_i} where v_i has valence 1) in S_0 lifts to X' . Consequently, this segment is a concatenation of paths in S_0 each of which is h -preperiodic, and therefore this segment is N -periodic (since it corresponds to an element of the group and so we have inverse). Thus every such segment a in S_0 is equivalent to some power $h^k(a)$ (note that there is a uniform bound for the powers) and hence for some k , h^k restricted to S_0 induces the identity on the quotient, since h is relative train track.

□

7 Stretching map

In this section we will see that we can define a homomorphism from the stabiliser of the lamination to \mathbb{R} .

Lemma 7.1. Suppose that $h : S \rightarrow S$ is an \mathcal{O} -map that represents $\psi \in \text{Out}(G, \mathcal{O})$. Then there exists a positive number $\lambda = \lambda(h, \Lambda)$ such that for every $\epsilon > 0$ there is $N > 0$ so that if L is a leaf segment of Λ of length $> N$, then $\left| \frac{\text{length}([h(L)])}{\text{length}(L)} - \lambda \right| < \epsilon$

Proof. We note that since f is IWIP, we have that the transition matrix $M = A(f)$ is irreducible (as it is every power of M) and therefore we can apply the Perron - Frobenius theorem to M , as a consequence we have that long leaf segments of Λ cross orbits of edges of T with frequencies close to those determined by the components of the PF eigenvector.

Now fix large k and then large l.s. are concatenation of l.s. of the form $f^k(e)$, for some edges of T , each orbit of edges with definite frequency. (For $k = 1$ this is the statement above, for $k > 1$ apply P.F theorem for f^k).

If M is large enough, then for any l.s. L with $\text{length}(L) > M$ we can think L as concatenation of l.s. of the form $f^k(e)$ (there are possible some shorts segments contained in the first and the final segment, which are not of this form but we can ignore them since their contribution in lengths is negligible).

Now let C be the bounded cancellation constant for $h : T \rightarrow T$, and let's denote $l_e = \text{len}(f^k(e))$, $l_e^h = \text{len}([h(f^k(e))])$, N_e be the number of occurrences of orbits of $f^k(e)$ in L and $N = \sum N_e$, then we have that $\frac{N_e}{N} \rightarrow r_e$, as $\text{len}(L) \rightarrow \infty$ (r_e is the PF component of the eigenvector that corresponds to e) by the PF theorem.

Note that the numbers N_e, l_e, l_e^h depends on k , so we define $a_k = \frac{\sum r_e l_e^h}{\sum r_e l_e}$. We have that $\text{len}(L) = \sum N_e l_e$ and by bounded cancellation lemma:

$$\frac{\sum N_e (l_e^h - 2C)}{\sum N_e l_e} \leq A_M = \frac{\text{len}([h(L)])}{\text{len}(L)} \leq \frac{\sum N_e l_e^h}{\sum N_e l_e} \quad (7.1)$$

and subdividing the sums by N we have that

$$\frac{\sum \frac{N_e}{N} l_e^h - 2C \frac{N_e}{N}}{\sum \frac{N_e}{N} l_e} \leq A_M = \frac{\text{len}([h(L)])}{\text{len}(L)} \leq \frac{\sum \frac{N_e}{N} l_e^h}{\sum \frac{N_e}{N} l_e} \quad (7.2)$$

where the term $2C \frac{N_e}{\sum N_e l_e}$ converges to 0 as $k \rightarrow \infty$ and as we noted above $\frac{N_e}{N} \rightarrow r_e$, as $\text{len}(L) \rightarrow \infty$. As a consequence, for every ϵ for large $k = k(\epsilon)$ and for large $M = M(\epsilon, k)$, $a_k - \epsilon \leq A_M \leq a_k + \epsilon$. Firstly, we send $M \rightarrow \infty$ and then for every $\epsilon > 0$ for large k ,

$$a_k - \epsilon \leq \liminf A_M \leq \limsup A_M \leq a_k + \epsilon \quad (7.3)$$

Therefore sending ϵ to 0, k to infinity, we have that, choosing a subsequence of a_k that converges to a ,

$$a \leq \liminf A_M \leq \limsup A_M \leq a \quad (7.4)$$

and therefore $\lim A_M = \liminf A_M = \limsup A_M = a$.

As consequence we have the requested property that there exists a positive number λ s.t. $\frac{\text{len}([h(L)])}{\text{len}(L)} \rightarrow \lambda$, as $\text{len}(L)$ is going to infinity. \square

Lemma 7.2. Using the notation as above and choosing any other representative h' of ψ , we have that $\lambda(h, \Lambda) = \lambda(h', \Lambda)$. In particular, the number doesn't depend on the representative but only on ψ .

Proof. Let h, h' be \mathcal{O} -maps which represent ψ as in the previous lemma. Therefore by the proposition 3.1 for any L , $[h(L)] = [h'(L)]$, up to bounded error that doesn't depend on L . Therefore for every L , $\text{len}([h(L)]) \leq \text{len}([h'(L)]) + C$, where C is positive fixed and as a consequence

$$\left| \frac{\text{len}([h(L)] - \text{len}([h'(L)]))}{\text{len}(L)} \right| \leq \frac{C}{\text{len}(L)} \rightarrow 0$$

for large $\text{len}(L)$.

Therefore since $\frac{\text{len}([h(L)])}{\text{len}(L)} \rightarrow \lambda(h, \Lambda)$ and $\frac{\text{len}([h'(L)])}{\text{len}(L)} \rightarrow \lambda(h', \Lambda)$, we have as a consequence $\lambda(h, \Lambda) = \lambda(h', \Lambda)$. \square

Lemma 7.3. Using the notation above we have that $\sigma : \text{Stab}(\Lambda) \rightarrow \mathbb{R}^+$, where $\sigma(\psi) = \lambda(h, \Lambda)$, is a well defined homomorphism.

Proof. Since we have that $\psi \in \text{Stab}(\Lambda)$, this means that $[h(\ell)]$ is a leaf (for any leaf ℓ) and as a consequence σ is a well defined map.

We will prove that σ is homomorphism.

So we have to prove that for any $\psi_1, \psi_2 \in \text{Stab}(\Lambda)$ it holds that $\sigma(\psi_1)\sigma(\psi_2) = \sigma(\psi_1\psi_2)$. We choose representatives h_1, h_2 of ψ_1, ψ_2 respectively, and by definitions $\frac{\text{len}([h_1(L)])}{\text{len}(L)} \rightarrow \sigma(\psi_1)$ and $\frac{\text{len}([h_2(L)])}{\text{len}(L)} \rightarrow \sigma(\psi_2)$. Moreover, h_1h_2 represents $\psi_1\psi_2$ (by previous lemma we can choose any representative).

Therefore since $\frac{\text{len}([h_1(h_2(L))])}{\text{len}(h_2(L))} \rightarrow \sigma(\psi_1\psi_2)$, for $\text{len}(L) \rightarrow \infty$ and $\frac{\text{len}([h_1(h_2(L))])}{\text{len}[h_2(L)]} = \frac{\text{len}([h_1[h_2(L)])]}{\text{len}[h_2(L)]} \frac{\text{len}([h_2(L)])}{\text{len}(L)}$ up to bounded error. But now sending $\text{len}(L)$ to infinity, it holds $\frac{\text{len}([h_1[h_2(L)])]}{\text{len}[h_2(L)]} \rightarrow \sigma(\psi_1)$ (as $\text{len}[h_2(L)]$ converges to infinity when $\text{len}(L) \rightarrow \infty$ and the fact that $[h_1[h_2(L)]]$ and $[h_1(h_2(L))]$ are in bounded distance and the bound doesn't depend on L).

Therefore by uniqueness of the limit, we have that $\sigma(\psi_1\psi_2) = \sigma(\psi_1)\sigma(\psi_2)$. \square

8 Kernel of the homomorphism

Now we investigate the properties of the kernel, We would like to prove that $\ker(\sigma)$ contains as subgroup of finite index the intersection of the stabiliser with the kernel of the action. But firstly, we aim to prove that the subgroup $\ker(\sigma)$ contains only non- exponentially growing automorphisms. We will prove it separately for irreducible and reducible automorphisms.

8.1 Reducible case

In the reducible case we will see that the automorphisms of the $\text{Stab}(\Lambda)$, have representatives of a very specific form. More specifically, every stratum except the top, is non-exponentially growing and moreover the representative restricted to each stratum is just a permutation of edges. Therefore we can calculate the value of σ , using only the top stratum if it is exponentially growing.

Proposition 8.1. If $\psi \in \text{Stab}(\Lambda)$ is exponentially growing and there exists some k s.t. ψ^k reducible, then $\psi \notin \text{Ker}(\sigma)$

Proof. Let $h : S \rightarrow S$ be a relative train track representative of ψ (we can change h with some power if it is necessary).

Firstly, we note that every stratum, except possibly the top one, is non-exponentially growing. This is true, since otherwise if some H_r is exponentially growing and $e \in H_r$ we have that the lengths of tightenings of h -iterates of e are arbitrarily long (by the train track properties) and they are l.s. (by definition of the stabiliser of the lamination), but this means that we have arbitrarily long segments contained in some proper subgraph (since $h(G_r) \subseteq G_r$), which is impossible as we have seen in 6.2.

Therefore if ψ is exponentially growing then we suppose, changing h with some iterate if it is necessary, that there exists H_0 which is union of strata, all of them are non-exponentially growing, h restricted to H_0 induces the identity in the quotient, and that the top stratum is exponentially growing, so if we have a leaf of the lamination and using the subgraph-overgraph decomposition of the leaf, it is implied that the lengths of long l.s. grow exponentially and in fact the actual value is the Perron-Frobenius eigenvalue that corresponds to the unique exponentially growing stratum. \square

8.2 Irreducible case

Now let's suppose that ψ is an IWIP. We have two cases and we will prove the theorem independently for automorphisms that have the *NGC* and the rest automorphisms that have the *GC* (the dichotomy is the same as in the free case, but for the automorphisms with *GC* we need arguments of different nature). We will prove again that the value of σ corresponds to the Perron - Frobenius eigenvalue of ψ (or ψ^{-1}).

Lemma 8.2. Let $h : S \rightarrow S$ be a train track map representing some irreducible $\psi \in \text{Out}(G, \mathcal{O})$.

Then for every $C > 0$ there is a number $M > 0$ such that if L is any path, then one of the following holds:

- (i) $[h^M(L)]$ contains a legal segment of length $> C$
- (ii) $[h^M(L)]$ has fewer illegal turns than L
- (iii) L is concatenation $x \cdot y \cdot z$, such that y is N-preperiodic and x, z have length $\leq 2C$ and at most one illegal turn.

Proof. Choose M to be a natural number that exceeds the number of inequivalent legal edge paths of length $\leq 2C$.

Now assume that L is a path such that the second statement fails, so $[h^M(L)]$ has the same number of illegal turns with L (since h is train track map, sends edges to legal paths and legal turns to legal turns so it is not possible the image of a path to have more illegal turns than the path). So each h -iteration of L amounts to iterating maximal legal subsegments of L and cancelling portions of adjacent ones.

If, in addition, the first fail as well, then each maximal legal segment (which has length $\leq C$) of L , except possibly the ones that contain the endpoints must have two iterates that after cancellation yield equivalent segments (otherwise we will have M equivalent legal segments of length $\leq C$, but this contradicts to the choice of M).

Therefore, we have that each segment contains a preperiodic point so that these points subdivide L as $x \cdot y_1 \cdot \dots \cdot y_m \cdot z$, and we have that this path satisfies the third statement. \square

Firstly we will prove a useful lemma for IWIP automorphisms which satisfy the property NGC and then we see that how we can use it for GC automorphisms.

Lemma 8.3. Let ψ, ψ^{-1} irreducible automorphisms (IWIP'S), $h : S \rightarrow S$ train track map representing ψ , $h' : S' \rightarrow S'$ representing ψ^{-1} and let's suppose that there is an integer m so that it is impossible to concatenate more than m N- periodic in S and in S' . Let $\tau : S \rightarrow S$, $\tau' : S' \rightarrow S'$, \mathcal{O} -maps.

Then for any $C > 0$ there are constants $N_0 > 0$ and L_0 such that if j is line or a path of length $\geq L_0$ and if j' the isometric immersion obtained from $[\tau j]$, then one of the following holds:

- (A) $[h^M(j)]$ contains a legal segment of length $> C$
- (B) $[h'^M(j')]$ contains a legal segment of length $> C$

Proof. Without loss, we may assume that C is larger than the critical constants for h and for h' . Let M be the larger of the two integers guaranteed by previous lemma applied to h, C and h', C . We will fix a large integer $s = s(h, h', \tau, \tau', M)$. Suppose that (A) does not hold with $N_0 = sM$. We will apply the previous lemma only to h^M -admissible segments (a segment $L \subseteq j$ so that $h^M(\partial L) \subseteq [h^M(j)]$).

By our assumption the first of the previous lemma doesn't hold. If we further restrict to segments L with $> m + 2$ illegal turns, then we can't have the third case either. So for such segments the second is always true. We can represent j as a concatenation of such segments of uniformly bounded length and the uniform bound does not depend on j , but only on h, h', τ, τ', M (since we will apply the same argument using $[\tau h(j)], h'$ instead of j, h respectively).

Say p is an upper bound to the number of illegal turns in each segment (there are finitely since they are of uniformly bounded length). Fix a with $\frac{p-1}{p} < a < 1$. For long enough segments L in j the ratio $\frac{\text{number of illegal turns in } [h^M(L)]}{\text{number of illegal turns in } L} < a$ (since the number of illegal turns in L than p and number of illegal turns in $[h^M(L)]$ is strictly less than the number of illegal turns in L).

By applying the same argument to $h^M(j)$ and then to $h^{2M}(j)$ etc, we see that for given $s > 0$ and long enough segments $L \subseteq j$ (the length depends on s as well), we have $\frac{\text{number of illegal turns in } [h^{sM}(L)]}{\text{number of illegal turns in } L} < a^s$, or else (A) holds with $N_0 = sM$. Since legal segments have length above by C and below by the length of the shortest edge (with the exception of the two containing the endpoints), the length can be compared with two inequalities to the number of illegal turns. Therefore if (A) fails, there exists a constant $A = A(h, C)$ with the property $\frac{\text{length}[h^{sM}(L)]}{\text{length}(L)} < Aa^s$. Similarly, we can use the same argument using $[\tau h^{sM}j]$ in place of j and with h' in place of h . If (B) fails as well, (with $N_0 = sM$) we reach a similar conclusion that $\frac{\text{length}[h'^{sM}\tau h^{sM}(L)]}{\text{length}[\tau h^{sM}(L)]} < Ba^s$ for some B depends only on h', C .

Firstly, we note that $h'^{sM}\tau h^{sM}, \tau$ are both \mathcal{O} -maps so they coincide to every path, except some bounded error near endpoints, in particular for long L , we have that the ratio of their lengths is bounded above by 2 and below by $1/2$. Therefore multiplying the above inequalities and changing $h'^{sM}\tau h^{sM}$ by τ we have the inequality :

$$\frac{\text{length}[\tau(L)]}{\text{length}[\tau h^{sM}(L)]} \frac{\text{length}[h^{sM}(L)]}{\text{length}(L)} < 2ABa^{2s}. \quad (8.1)$$

On the other hand, $\frac{\text{length}[h^{sM}(L)]}{\text{length}(\tau h^{sM}L)} \frac{\text{length}[\tau(L)]}{\text{length}(L)} > \frac{1}{2\text{Lip}(\tau)\text{Lip}(\tau')}$ using again that $\tau'\tau$ and the identity are both \mathcal{O} -maps as above.

But sending s to infinity we have a contradiction, since $a < 1$. \square

Geometric Case: In the proof of the previous lemma we have used the property that there is an integer m so that it is impossible to concatenate more than m N- periodic paths in j (and the iterates $[h^M(j)]$) and the same is true for j' (and the iterates $[h'^M(j')]$). The previous lemma is true for NGC automorphisms for every j . But if we apply this when j is some leaf of the lamination and $h \in \text{Stab}(\Lambda)$, we can prove that this always the case.

Lemma 8.4. If ℓ is some leaf of the lamination, then there is an integer m so that it is not possible for ℓ to contain a concatenation of m subpaths that each of them is N-periodic.

Proof. Choose $f : T \rightarrow T$, stable train track representative (this is possible by [5], since N-periodic paths correspond to Nielsen periodic paths in the quotient or see [23] for a different approach), then there is exactly one path in $\Gamma = G/T$ in which every (indivisible) N-periodic path projects. We suppose that there is no bound in the number of concatenation of INP in ℓ . So by quasiperiodicity we have that every leaf segment is contained in some concatenation of equivalent paths of the form $P_1 P_2 \dots P_n$ (where every P_i is a path that projects to the loop P). But then the subgroup that is constructed by the graph of groups corresponding to this loop (see the section 2.6. of the preliminaries), carries the lamination and therefore has finite index (by 6.2) in G , which is impossible. \square

Therefore the lemma 8.3 is true, in this case, if we restrict to $h \in \text{Stab}(\Lambda)$ and ℓ some leaf of the lamination.

Definition 8.5. We say that a sequence $\{\Lambda_i\}$ of irreducible laminations in \mathcal{IL} if for some (any) tree H every leaf segment of Λ in S -coordinates is a leaf segment of Λ_i in S -coordinates for all but finitely many i .

Proposition 8.6. Let $\Lambda = \Lambda_\phi^+ \in \mathcal{IL}$ and let $\psi \in \text{Aut}(G, \mathcal{O})$ which is an IWIP. Suppose that $\psi \in \text{Stab}(\Lambda)$, then $\Lambda = \Lambda_\psi^+$ or $\Lambda = \Lambda_\psi^-$.

We note again that if a segment contains a legal segment with length larger than C_{crit} then the length of reduced iterates converge to infinity.

Proof. In the non-geometric case:

Using the notation of the previous lemmas. Let ℓ be a leaf of Λ in the S -coordinates. We apply the lemma to $[h^K \ell]$ with $K > 0$ and C larger the critical constants of h and h' . If for some $K > 0$ (A) holds, then it follows from quasiperiodicity that the forward iterates weakly converges to Λ_ψ^+ , since we have that the length of reduced images converges to infinity and so we have arbitrarily long legal segments and the quasiperiodicity implies that some translation of every leaf segment is finally contained in the reduced images.

The remaining possibility is that $[\tau h^K \ell]$ contains an S' legal segment of length $> C$ for all $K > 0$. But this means that $[\tau \ell]$ which equals to $[h'^K \tau h^K \ell]$ up to bounded error, contains an arbitrarily high h' -iterate of a legal segment and quasiperiodicity now implies that $\Lambda = \Lambda_h^-$.

Now in the geometric case, we use the same argument but only for $h \in \text{Stab}(\Lambda)$ and we have the same result that $\Lambda = \Lambda_h^\pm$ \square

Note that we have proved that for automorphisms with the property NGC, it is true for every IWIP ψ (relative to \mathcal{O}) either the forward ψ -iterates of Λ weakly converges to Λ_ψ^+ or $\Lambda = \Lambda_\psi^-$.

Corollary 8.7. If $\psi \in \text{Stab}(\Lambda)$ is exponentially growing, then $\psi \notin \text{Ker}(\sigma)$

Proof. For reducible automorphisms, we have already proved it in 8.1. For irreducible ones, we have by the previous proposition that $\Lambda = \Lambda_\psi^+$ (changing ψ with ψ^{-1} , if it necessary) and so we can choose $f = h$, where h is the train track representative of ψ , in the proof of 7.1, and then $\sigma(\psi)$ is obviously equal to the Perron - Frobenius eigenvalue which is greater than 1, since ψ is exponentially growing(it is an IWIP). \square

8.3 Discreteness of the Image

We will prove that the image of the homomorphism σ is discrete and therefore we can see σ as a homomorphism $\sigma : \text{Stab}(\Lambda) \rightarrow \mathbb{Z}$.

Lemma 8.8. $\sigma(\text{Stab}(\Lambda))$ is a discrete set.

Proof. This is true since by the proofs of the propositions (8.1, 8.7), every $\sigma(\psi)$ other than 1, occurs as the Perron- Frobenius eigenvalue for an irreducible integer matrix of uniformly bounded size. It is well known then that the set of such numbers form a discrete set and as a consequence $\sigma(\text{Stab}(\Lambda))$ is an infinite discrete subset of \mathbb{R} and is hence isomorphic to \mathbb{Z} . \square

9 Main Results

In this section, we will state and prove the main theorems. We use the same notation as in the sections above.

Lemma 9.1. Let $h : S \rightarrow S$ be a relative train track representative of $\psi \in \text{Ker}(\sigma)$. Then there is some k such that h^k induces the identity on $G \setminus S$. Moreover, there are appropriate representatives of orbits of non-free vertices v_1, \dots, v_q , such that $h(v_i) = v_i$. Finally, if $\psi \in \text{Ker}(\sigma) \cap \text{Out}(G, \{G_i\}^t)$ then ψ is an automorphism of finite order.

Proof. Let $\psi \in \text{Ker}(\sigma)$ and $h' : S \rightarrow S$ be a RTT train track representative of ψ .

By 8.7, possibly after changing ψ with some iterate ψ^k , we can suppose that there is a relative train track representative, $h^k = h : S \rightarrow S$ and a maximal proper h -invariant G -subgraph S_0 of S (we denote by H_0 the quotient S_0/G) s.t. the restriction of h on S_0 induces the identity in H_0 . For the top stratum we can suppose that it contains a single edge e and that $h(e) = ea$, where a is some segment of S_0 (since it is non-exponentially growing). But then since h is a relative train track and $h \in \text{Stab}(\Lambda)$, we have that h -iterates of e produces arbitrarily long segments of the lamination that are contained in S_0 which contradicts quasiperiodicity, except if the leaf of the lamination is of the form (in the quotient, so every a, e correspond to orbits):

$$\dots ea^{b_{-1}}e^{-1}x_{-1}ea^{b_0}e^{-1}x_0ea^{b_1}e^{-1}x_1ea^{b_2}e^{-1}\dots$$

for some integers b_i and x_i are contained in S_0 (or H_0 in the quotient). In this case, the lamination is carried by the subgroup which is the fundamental group of the graph of groups which consists of the disjoint union of two graph of groups corresponding to H_0 (which contains all the non-free vertices with full stabilisers) that are joined by an edge

corresponding to e . But by 6.2, this leads to a contradiction since it is obvious that this subgroup is not of finite index (and by construction it contains the full stabilisers of vertices). Therefore we have that $h(e) = e$ and then h induces also the identity on $\Gamma - H_0$ ($\Gamma = G/T$) and so on Γ .

Now suppose $h(e_1) = e_1, h(e_2) = g_0 e_2$, where $g_0 \in G_v$ as above, where $e_1 e_2$ is a legal path. Then since h is a isometry we have that we will have as leaf segments of the form $e_1 g_n e_2$ where $g_n = \psi^n(g_1)$. But since there are finitely many inequivalent paths of a specific length, we can get that after passing some power if needed, that there is some $g \in G_v$ such that $h(g e_2) = g e_2$ and after changing the fundamental domain (in particular, e_2 with $g e_2$), we have that h fixes pointwise the fundamental domain. Since this can be done for every vertex we have that we can suppose that every edge of the fundamental domain is fixed by h (after possibly passing to some power). Therefore h is an automorphism that sends a path of the form $g_1 e_1, \dots, g_m e_m$ to the path $\psi(g_1) e_1, \dots, \psi(g_m) e_m$ where $g_i, \psi(g_i) \in G_{\partial(e_i)}$, and as a consequence h depends only on the induced automorphisms on each G_i .

From the discussion above, if we assume also that $\psi \in' Out((G, \{G_i\}^t)$ the induced automorphism on each G_i is the identity and therefore h is the identity, which implies that there is some k such that ψ^k is represented by the identity and that means that ψ^k is the identity. \square

As a consequence, let's consider the subgroup $A = Out((G, \{G_i\}^t) \cap Ker(\sigma)$ which, by the previous lemma, is periodic. Then $Stab(\Lambda)/A$ has a normal subgroup $B = Ker(\sigma)/A$, which is isomorphic to a subgroup $\bigoplus_{i=1}^p A_i$ of $\bigoplus_{i=1}^p Out(G_i)$ and $(Stab(\Lambda)/A)/B$ is an infinite cyclic group. If we further assume that $Out(G)$ is virtually torsion free (as for example in the free and the relative free case), then we have that $Stab(\Lambda)$ has a (torsion free) finite index subgroup A' . Then $A' \cap A = 1$ (since A is torsion free and A' periodic) and so that $A'/B \cap A'$ is isomorphic to \mathbb{Z} , where $A' \cap B$ is isomorphic to a subgroup of $\bigoplus_{i=1}^p Out(G_i)$.

Finally, let's also assume that every $Out(G_i)$ is finite, then we get that A' is actually isomorphic to \mathbb{Z} , so we have exactly the same result as in the classical case of the free group, and more precisely $Stab(\Lambda)$ is virtually \mathbb{Z} . As conclusion of the discussion above, we get:

- Theorem 9.2.** (i) There is a normal periodic subgroup A of $Stab(\Lambda)$, such that the group $Stab(\Lambda)/A$ has a normal subgroup B isomorphic to subgroup of $\bigoplus_{i=1}^q Out(G_i)$ and $(Stab(\Lambda)/A)/B$ is isomorphic to \mathbb{Z} .
- (ii) Let's also suppose that $Out(G)$ is virtually torsion free. Then $Stab(\Lambda)$ has a (torsion free) finite index subgroup K such that K/B' is isomorphic to \mathbb{Z} , where B' is a normal subgroup of K isomorphic to subgroup of $\bigoplus_{i=1}^q Out(G_i)$.
- (iii) Finally, if we further suppose that every $Out(G_i)$ is finite, then $Stab(\Lambda)$ is virtually (infinite) cyclic.

A direct corollary of the previous theorem is the following. Let's denote $C(\phi)$ the relative centraliser of ϕ in $Out(G, \mathcal{O})$. As we have seen, $C(\phi)$ is a subgroup of $Stab(\Lambda)$ and so we get:

- Theorem 9.3.** (i) There is a normal periodic subgroup A_1 of $C(\phi)$, such that the group $C(\phi)/A_1$ has a normal subgroup B_1 isomorphic to a subgroup of $\bigoplus_{i=1}^q Out(G_i)$ and $(C(\phi)/A_1)/B_1$ is isomorphic to \mathbb{Z} .
- (ii) Let's also suppose that $Out(G)$ is virtually torsion free. Then $C(\phi)$ has a (torsion free) finite index subgroup K'_1 such that K'_1/B'_1 is isomorphic to \mathbb{Z} , where B'_1 is a normal subgroup of K'_1 isomorphic to subgroup of $\bigoplus_{i=1}^q Out(G_i)$.
- (iii) Finally, if we further suppose that every $Out(G_i)$ is finite, then $C(\phi)$ is virtually (infinite) cyclic.

Note that in the case in which \mathcal{O} corresponds to the Grushko decomposition of G , we have that the previous theorem generalises the theorem in the classical case that the centraliser of an IWIP (for f.g. free groups with the absolute notion of irreducibility) is virtually cyclic since there are no G_i 's and so the factor automorphisms are trivial in the free case. Additionally, we can take also relative results for the free and for the general case. This is possible since we can use the fact

that every automorphism is irreducible relative to some appropriate space.

Moreover, note that if ϕ doesn't commute with the automorphisms of the free factors then $C(\phi)$ is virtually cyclic. But as we will see, in the general case there are examples that this is not true. In particular, we can find centralisers of IWIP automorphisms (relative to some space) which contain big subgroups and as a consequence they are not virtually cyclic.

In fact, we can get something stronger than the previous theorem. Remember that if G is a group and H is a subgroup of G , the commensurator (or virtual normalizer) of H in G is the subgroup $Comm_G(H) =: \{g \in G \mid [H : H \cap g^{-1}Hg] < \infty, \text{ and } [g^{-1}Hg : H \cap g^{-1}Hg] < \infty\}$. Here we have that the commensurator $Comm_{Out(G, \mathcal{O})}(\phi)$ contains $C_{Out(G, \mathcal{O})}(\phi)$ for every automorphism ϕ . But for every IWIP ϕ the subgroup $Comm_{Out(G, \mathcal{O})}(\phi)$ stabilises the lamination, since for $\psi \in Comm_{Out(G, \mathcal{O})}(\phi)$ there are n, m such that $\psi\phi^m\psi^{-1} = \phi^n$, we get a similar statement as above for commensurators of IWIP automorphisms instead of centralisers.

Now let's give an example of a relative IWIP which has (relative) centraliser which fails to be virtually cyclic.

Example 9.4. *As in the introduction, we fix the free product decomposition $G = G_1 * \langle a \rangle * \langle b \rangle$, where a, b are of infinite order and we denote by $F_2 = \langle a \rangle * \langle b \rangle$ the "free part". Then in the corresponding outer space $\mathcal{O}(F_4, G_1, F_2)$, which we denote by \mathcal{O} . In each tree $T \in \mathcal{O}$ there is exactly one non free vertex v_1 s.t $G_{v_1} = G_1$. Then we define the outer automorphism ϕ , which satisfies $\phi(g) = g$ for every $g \in G_1$, $\phi(a) = bg_1$, $\phi(b) = ab$ for some non-trivial $g_1 \in G_1$, then $\phi \in Out(G, \mathcal{O})$ is an IWIP relative to \mathcal{O} . But every factor automorphism of G_1 that fixes g_1 commutes with ϕ and therefore $C(\phi)$ contains the subgroup A of $Aut(G_1)Inn(G)$ that fixes g_1 . So the centraliser is not virtually cyclic if A is sufficiently big.*

We will prove that ϕ is an IWIP relative to \mathcal{O} . Firstly, note that there are no ϕ -invariant free factor systems of the form $\{[G_1], \langle b \rangle\}$ or $\langle G_1, b \rangle$ that contain the free factor system $\{[G_1]\}$. So the only possible case is the case where there is a ϕ -invariant free factor system of the form $\{[G_1], \langle x, y \rangle\}$. Using the fact that we have two free factors, we can assume that the free factors G_1 and $\langle x, y \rangle$ are ac-

tually ϕ -invariant. Therefore after possibly changing the basis we can suppose that the projection map from G to $G / \langle\langle G_1 \rangle\rangle = \langle a, b \rangle$ sends x, y to a, b , respectively. Moreover, we can see that $x = am$, $y = bn$, where $m, n \in \langle\langle G_1 \rangle\rangle$. By the relations, $\phi(G_1) = G_1$ and $\phi(\langle x, y \rangle) = \langle x, y \rangle$, we have that ϕ induces the identity on G_1 (after possibly conjugacy with an element of G_1). In the first case, we can see that $\phi(x) = xy$ and $\phi(y) = x$. Then we get the identities $(am)(bn) = ab(\phi(m))$ and $am = ag_1\phi(n)$. By combining these together, we have that $mbg_1^{-1}\phi^{-1}(m) = b\phi(m)$ which easily leads to a contradiction to the fact that $m \in \langle\langle G_1 \rangle\rangle$. Similarly, we get a contradiction in the second case. Therefore there is no such a ϕ -invariant free factor system.

In the case that G_1 is isomorphic to F_3 and g_1 an element of its free basis, we have that $C(\phi)$ contains a subgroup which is isomorphic to $\text{Aut}(F_2)\text{Inn}(G)$.

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